Limits and Continuity

Bernd Schröder

Introduction

Introduction

1. The idea for the limit is the attempt to answer the question "What happens when the input of our function gets close to a number at which the function cannot be evaluated?"

Introduction

1. The idea for the limit is the attempt to answer the question "What happens when the input of our function gets close to a number at which the function cannot be evaluated?" For example, consider $D(h) = \dfrac{f(z_0+h)-f(z_0)}{h}$.

Introduction

1. The idea for the limit is the attempt to answer the question "What happens when the input of our function gets close to a number at which the function cannot be evaluated?" For example, consider $D(h) = \dfrac{f(z_0+h) - f(z_0)}{h}$.
2. Limits also help answer the question what happens when the absolute value of the input gets large.

Introduction

1. The idea for the limit is the attempt to answer the question "What happens when the input of our function gets close to a number at which the function cannot be evaluated?" For example, consider $D(h) = \frac{f(z_0+h) - f(z_0)}{h}$.
2. Limits also help answer the question what happens when the absolute value of the input gets large.
3. Continuous functions are functions for which limits are computed "by plugging in the number."

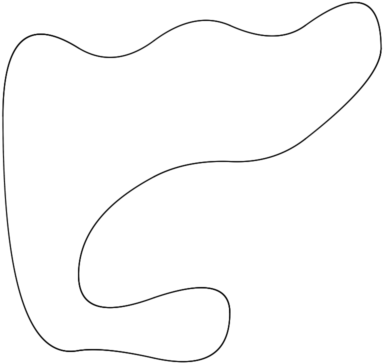
Introduction

1. The idea for the limit is the attempt to answer the question "What happens when the input of our function gets close to a number at which the function cannot be evaluated?" For example, consider $D(h) = \frac{f(z_0+h)-f(z_0)}{h}$.
2. Limits also help answer the question what happens when the absolute value of the input gets large.
3. Continuous functions are functions for which limits are computed "by plugging in the number."
4. Unlike for functions on the real line, the two-dimensional nature of $\mathbb{C}$ makes it sensible to first consider the properties of the domain.

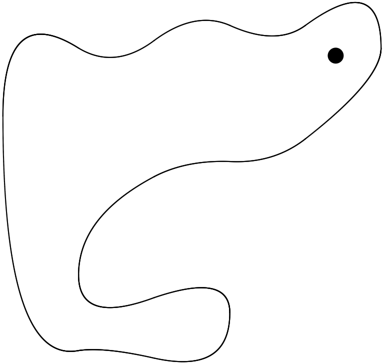
Definition.

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.

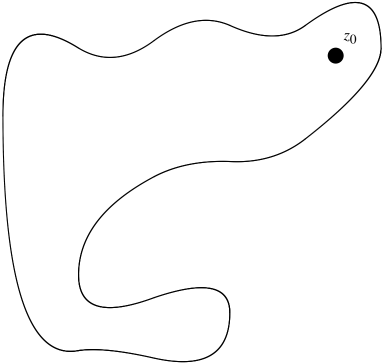
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.



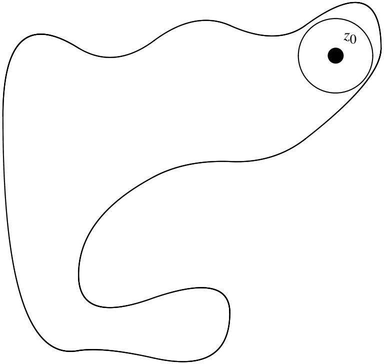
Definition. *Let $z_0 \in \mathbb{C}$ and let $\varepsilon > 0$. Then the* **ε-neighborhood** *of z_0 is the set of all $z \in \mathbb{C}$ so that $|z - z_0| < \varepsilon$.*



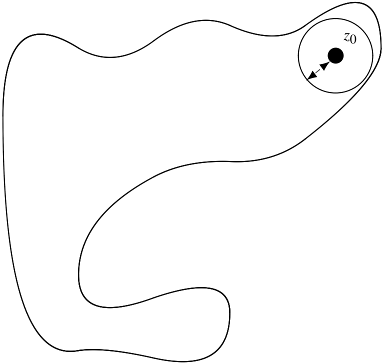
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.



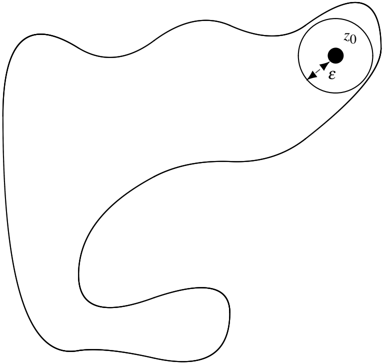
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.



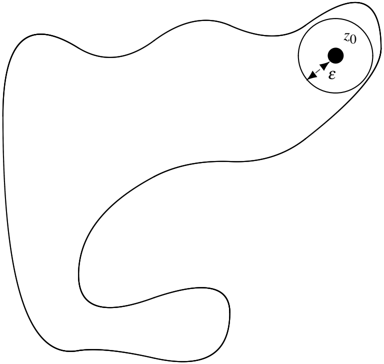
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.



Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $|z - z_0| < \varepsilon$.



Definition. *Let $z_0 \in \mathbb{C}$ and let $\varepsilon > 0$. Then the* **ε-neighborhood** *of z_0 is the set of all $z \in \mathbb{C}$ so that $|z - z_0| < \varepsilon$.*

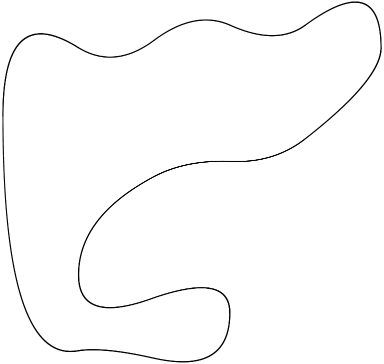


The idea is to encode "closeness" with ε-neighborhoods.

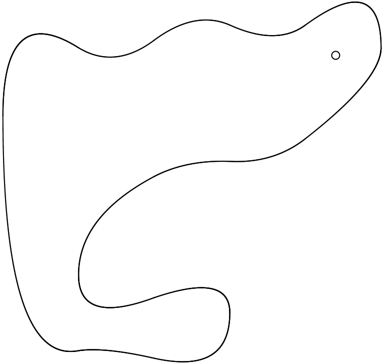
Definition.

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$*. Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

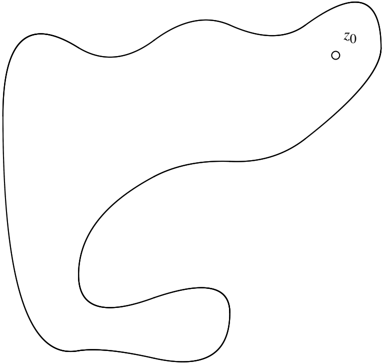
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.



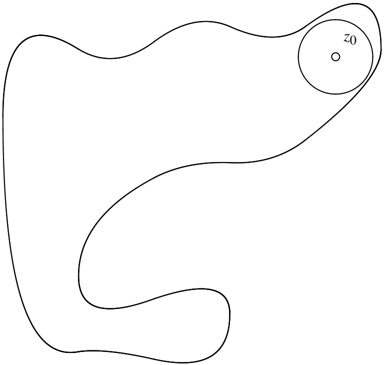
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.



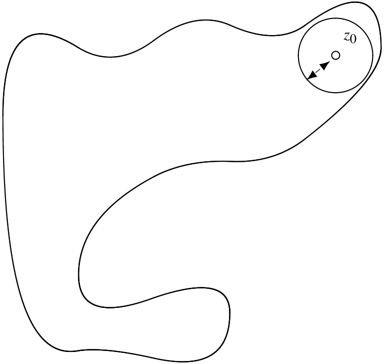
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.



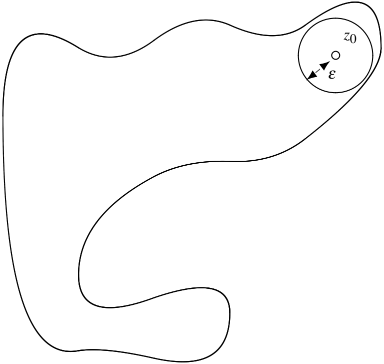
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$*. Then the* **deleted** ε**-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$*.*



Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$*. Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.



Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.



Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Limits are considered on deleted ε-neighborhoods

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Limits are considered on deleted ε-neighborhoods: The function is defined near

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Limits are considered on deleted ε-neighborhoods: The function is defined near, *but not at*

Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Limits are considered on deleted ε-neighborhoods: The function is defined near, *but not at*, a point z_0.

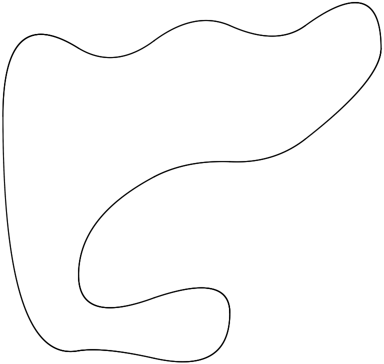
Definition. *Let* $z_0 \in \mathbb{C}$ *and let* $\varepsilon > 0$. *Then the* **deleted ε-neighborhood** *of* z_0 *is the set of all* $z \in \mathbb{C}$ *so that* $0 < |z - z_0| < \varepsilon$.

Limits are considered on deleted ε-neighborhoods: The function is defined near, *but not at*, a point z_0. We want to use the behavior in the deleted ε-neighborhood to find out what happens at z_0.

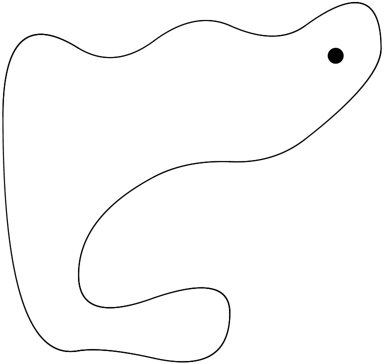
Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** if and only if S contains an ε-neighborhood of z_0.

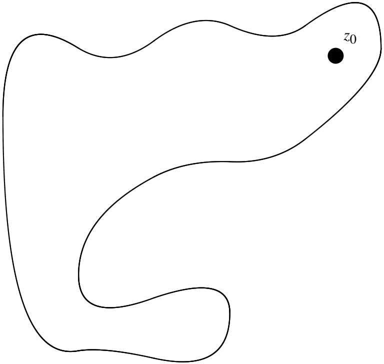
Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** if and only if S contains an ε-neighborhood of z_0.



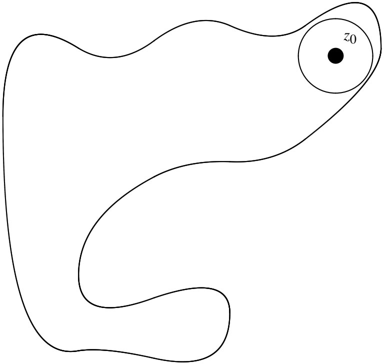
Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** if and only if S contains an ε-neighborhood of z_0.



Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** if and only if S contains an ε-neighborhood of z_0.



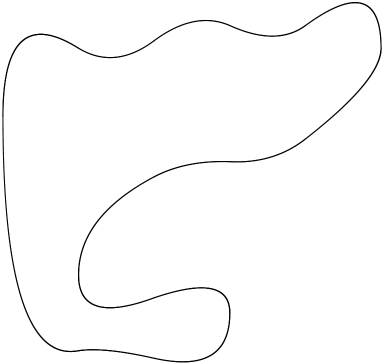
Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \in S$ is called an **interior point** if and only if S contains an ε-neighborhood of z_0.



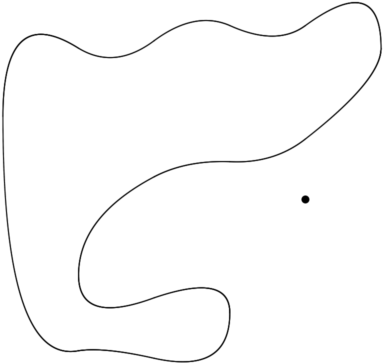
Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \notin S$ is called an **exterior point** if and only if there is an ε-neighborhood of z_0 that contains no points of S.

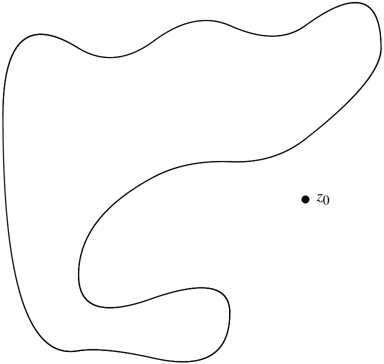
Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \notin S$ is called an **exterior point** if and only if there is an ε-neighborhood of z_0 that contains no points of S.



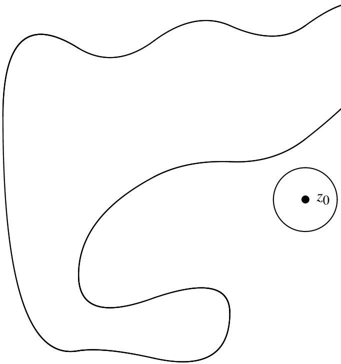
Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \notin S$ is called an **exterior point** if and only if there is an ε-neighborhood of z_0 that contains no points of S.



Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \notin S$ is called an **exterior point** if and only if there is an ε-neighborhood of z_0 that contains no points of S.



Consider a set $S \subseteq \mathbb{C}$. A point $z_0 \notin S$ is called an **exterior point** if and only if there is an ε-neighborhood of z_0 that contains no points of S.

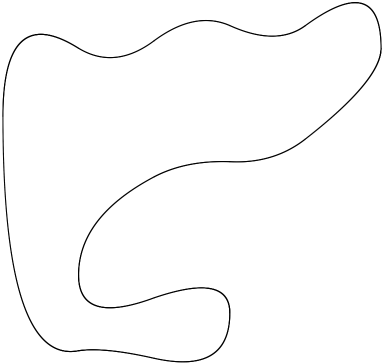


Consider a set $S \subseteq \mathbb{C}$.

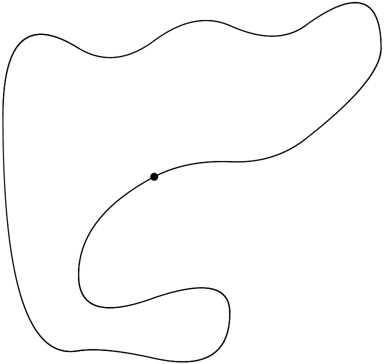
Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**.

Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**. That is, if z_0 is a boundary point, then every neighborhood of z_0 contains points that are in S and points that are not in S.

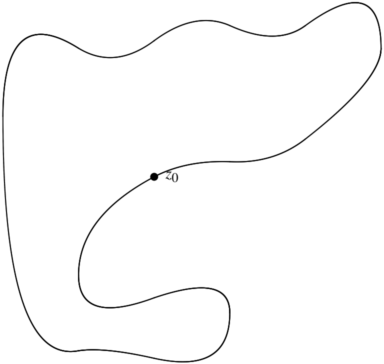
Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**. That is, if z_0 is a boundary point, then every neighborhood of z_0 contains points that are in S and points that are not in S.



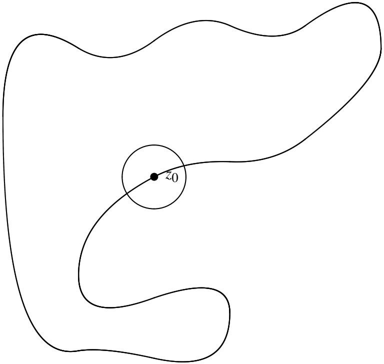
Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**. That is, if z_0 is a boundary point, then every neighborhood of z_0 contains points that are in S and points that are not in S.



Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**. That is, if z_0 is a boundary point, then every neighborhood of z_0 contains points that are in S and points that are not in S.



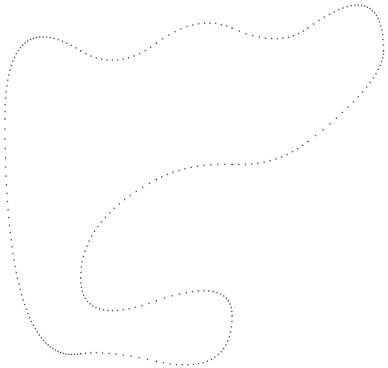
Consider a set $S \subseteq \mathbb{C}$. A point that is neither an interior nor an exterior point is called a **boundary point**. That is, if z_0 is a boundary point, then every neighborhood of z_0 contains points that are in S and points that are not in S.



Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A set is called **open** if and only if it contains none of its boundary points.

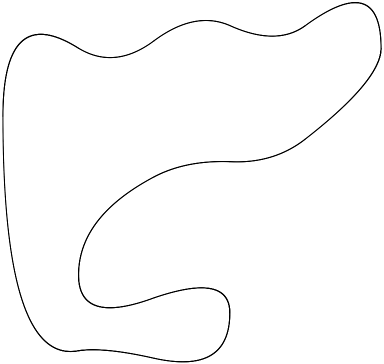
Consider a set $S \subseteq \mathbb{C}$. A set is called **open** if and only if it contains none of its boundary points.



Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A set is called **closed** if and only if it contains all of its boundary points.

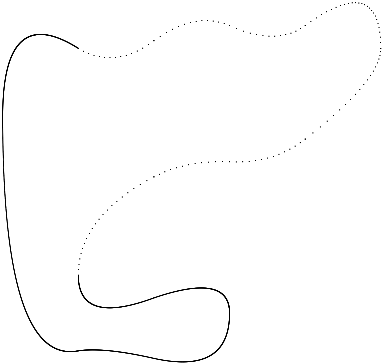
Consider a set $S \subseteq \mathbb{C}$. A set is called **closed** if and only if it contains all of its boundary points.



Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A set can be neither open nor closed.

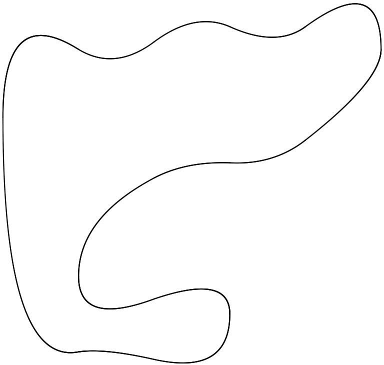
Consider a set $S \subseteq \mathbb{C}$. A set can be neither open nor closed.



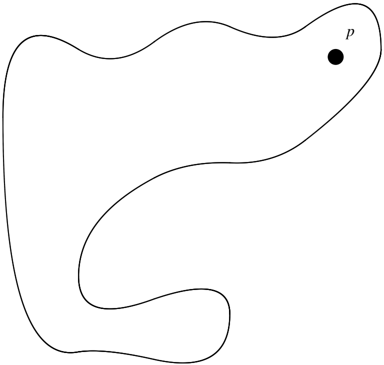
Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.

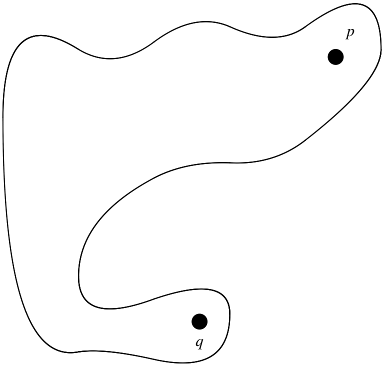
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



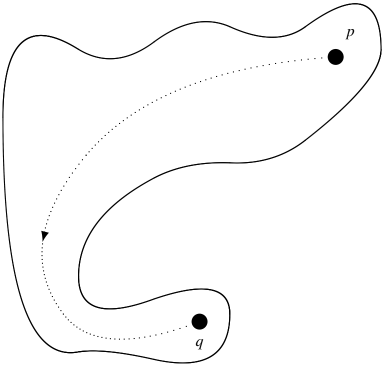
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



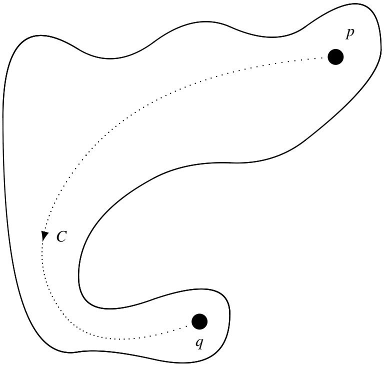
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



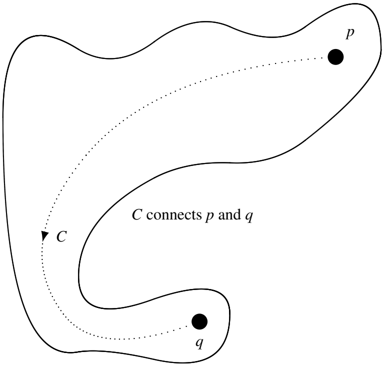
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



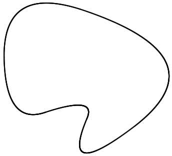
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.



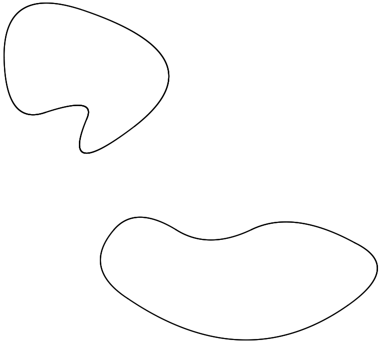
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path.

Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.

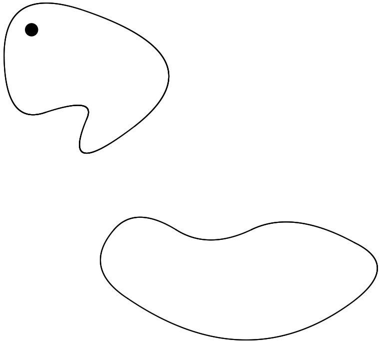
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



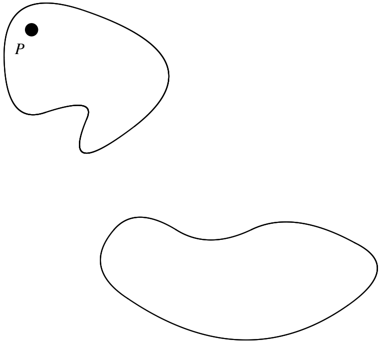
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



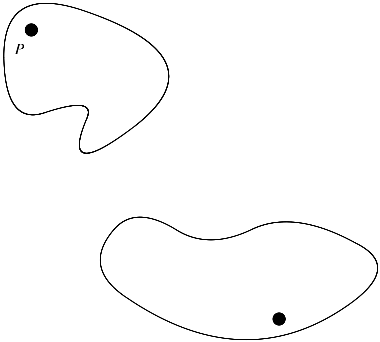
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



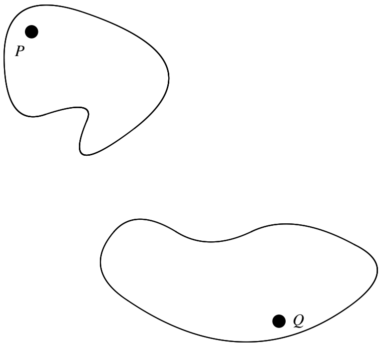
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



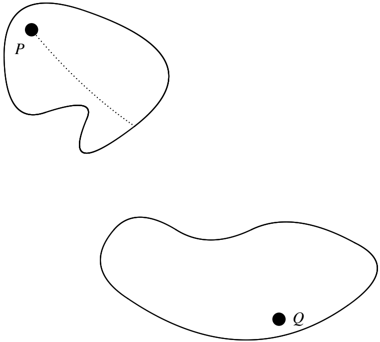
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



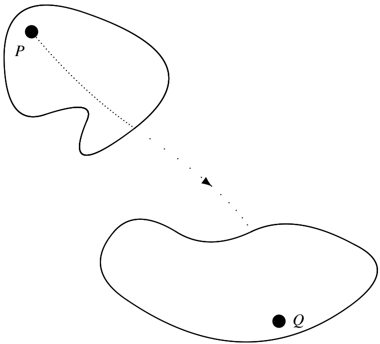
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



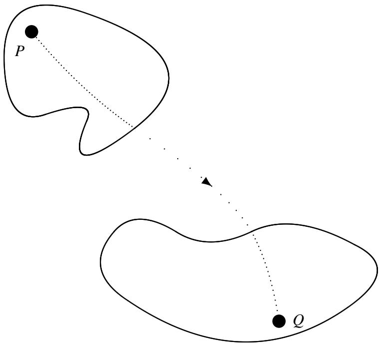
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



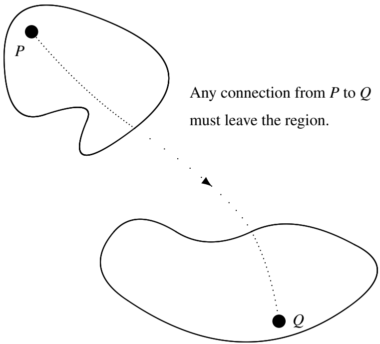
Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



Consider a set $S \subseteq \mathbb{C}$. An open set S is called **(pathwise) connected** if and only if from any point in S to any other point in S there is a (polygonal) path. Otherwise, the set is not pathwise connected.



Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A nonempty, open, connected set is called a **domain**.

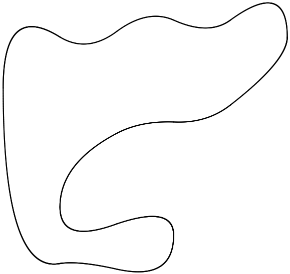
Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A domain with some, none or all of its boundary points is called a **region**.

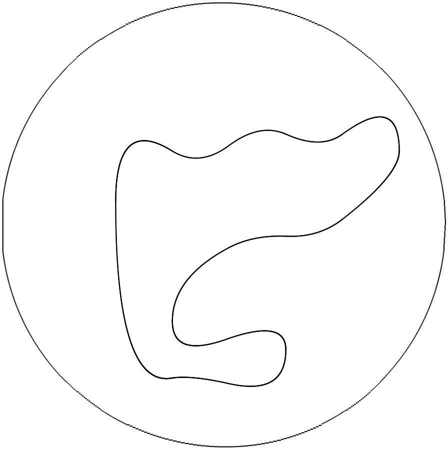
Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. S is **bounded** if and only if it lies inside some circle $|z| = r$.

Consider a set $S \subseteq \mathbb{C}$. S is **bounded** if and only if it lies inside some circle $|z| = r$.



Consider a set $S \subseteq \mathbb{C}$. S is **bounded** if and only if it lies inside some circle $|z| = r$.



Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. If S is not bounded, it is called **unbounded**.

Consider a set $S \subseteq \mathbb{C}$.

Consider a set $S \subseteq \mathbb{C}$. A point z_0 is called an **accumulation point** of S if and only if every deleted neighborhood of z_0 contains a point in S.

Informal idea for the limit.

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition.

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function.

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω.*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*

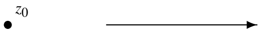
-

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*

Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



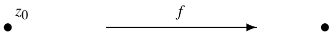
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*

$$\bullet^{z_0} \qquad \xrightarrow{\quad f \quad}$$

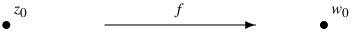
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



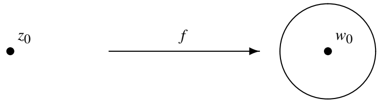
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



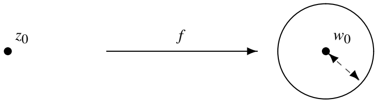
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



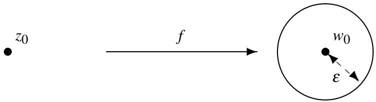
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



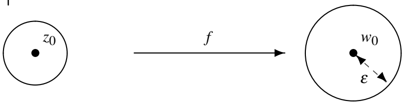
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



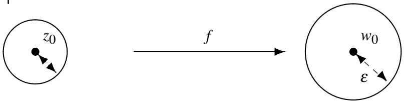
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



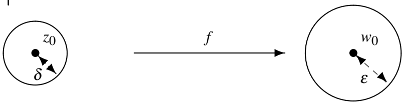
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $|f(z) - w_0| < \varepsilon$.*



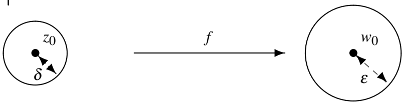
Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



Informal idea for the limit. w_0 is the limit of f at z_0 if and only if for all z sufficiently close to z_0 we have that $f(z)$ is close to w_0.

Definition. ε-δ formulation of the limit of a function. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z) - w_0\right| < \varepsilon$.*



Notation. $\lim_{z \to z_0} f(z) = w_0$.

Limits are Unique

Limits are Unique

Theorem.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω.*

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\left|w_0 - w_1\right|$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\left|w_0 - w_1\right| = \left|w_0 - f(z) + f(z) - w_1\right|$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\big|f(z) - w_0\big| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\big|f(z) - w_1\big| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$|w_0 - w_1| \quad = \quad \big|w_0 - f(z) + f(z) - w_1\big| \leq \big|w_0 - f(z)\big| + \big|f(z) - w_1\big|$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \left|w_0 - f(z) + f(z) - w_1\right| \leq \left|w_0 - f(z)\right| + \left|f(z) - w_1\right| \\ &< \frac{\varepsilon}{2} \end{aligned}$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\big|f(z) - w_0\big| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\big|f(z) - w_1\big| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \big|w_0 - f(z) + f(z) - w_1\big| \leq \big|w_0 - f(z)\big| + \big|f(z) - w_1\big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \left|w_0 - f(z) + f(z) - w_1\right| \leq \left|w_0 - f(z)\right| + \left|f(z) - w_1\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \left|w_0 - f(z) + f(z) - w_1\right| \leq \left|w_0 - f(z)\right| + \left|f(z) - w_1\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, the difference between w_0 and w_1 is arbitrarily small.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \left|w_0 - f(z) + f(z) - w_1\right| \leq \left|w_0 - f(z)\right| + \left|f(z) - w_1\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, the difference between w_0 and w_1 is arbitrarily small. Hence we must have $w_0 = w_1$.

Limits are Unique

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. If w_0 and w_1 are limits of f at z_0, then $w_0 = w_1$.*

Proof. Let $\varepsilon > 0$. Because w_0 and w_1 are limits of f at z_0, there is a $\delta_0 > 0$ so that for all $|z - z_0| < \delta_0$ we have $\left|f(z) - w_0\right| < \frac{\varepsilon}{2}$ and there is a $\delta_1 > 0$ so that for all $|z - z_0| < \delta_1$ we have $\left|f(z) - w_1\right| < \frac{\varepsilon}{2}$. Let $|z - z_0| < \min\{\delta_0, \delta_1\}$. Then

$$\begin{aligned} |w_0 - w_1| &= \left|w_0 - f(z) + f(z) - w_1\right| \leq \left|w_0 - f(z)\right| + \left|f(z) - w_1\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, the difference between w_0 and w_1 is arbitrarily small. Hence we must have $w_0 = w_1$. ■

Example.

Example. $\lim_{z \to i} z^2 = -1$

Example. $\lim_{z \to i} z^2 = -1$

Scratch work

Example. $\lim_{z \to i} z^2 = -1$

Scratch work: $\varepsilon > \left| z^2 - (-1) \right|$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2 - (-1)\right| = \left|z^2 + 1\right|$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$.

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2 - (-1)\right| = \left|z^2 + 1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2 - (-1)\right| = \left|z^2 + 1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$ and the term $|z+i|$will stay bounded as $z \to i$.

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument: Let $\varepsilon > 0$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$.

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$ and the term $|z+i|$will stay bounded as $z \to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$. Then for all $|z-i| < \delta$ we have

$$\left|z^2-(-1)\right|$$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument: Let $\varepsilon>0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3},1\right\}$. Then for all $|z-i|<\delta$ we have

$$\left|z^2-(-1)\right| \quad = \quad \left|z^2+1\right|$$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$ and the term $|z+i|$will stay bounded as $z \to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$. Then for all $|z-i| < \delta$ we have

$$\begin{aligned} \left|z^2-(-1)\right| &= \left|z^2+1\right| \\ &= |z+i|\,|z-i| \end{aligned}$$

Example. $\lim_{z \to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2 - (-1)\right| = \left|z^2 + 1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$ and the term $|z+i|$will stay bounded as $z \to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$. Then for all $|z-i| < \delta$ we have

$$\begin{aligned} \left|z^2 - (-1)\right| &= \left|z^2 + 1\right| \\ &= |z+i|\,|z-i| \\ &\leq \big(|z-i| + |2i|\big)\,|z-i| \end{aligned}$$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$. Then for all $|z-i| < \delta$ we have

$$\begin{aligned}
\left|z^2-(-1)\right| &= \left|z^2+1\right| \\
&= |z+i|\,|z-i| \\
&\leq \left(|z-i|+|2i|\right)|z-i| \\
&< 3
\end{aligned}$$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z\to i$ and the term $|z+i|$will stay bounded as $z\to i$.

Clean argument: Let $\varepsilon>0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3},1\right\}$. Then for all $|z-i|<\delta$ we have

$$\begin{aligned}\left|z^2-(-1)\right| &= \left|z^2+1\right| \\ &= |z+i|\,|z-i| \\ &\le \big(|z-i|+|2i|\big)\,|z-i| \\ &< 3\cdot\frac{\varepsilon}{3}\end{aligned}$$

Example. $\lim_{z\to i} z^2 = -1$

Scratch work: $\varepsilon > \left|z^2-(-1)\right| = \left|z^2+1\right| = |z+i|\,|z-i|$. The term $|z-i|$ can be made small as $z \to i$ and the term $|z+i|$will stay bounded as $z \to i$.

Clean argument: Let $\varepsilon > 0$ and choose $\delta := \min\left\{\frac{\varepsilon}{3}, 1\right\}$. Then for all $|z-i| < \delta$ we have

$$\begin{aligned} \left|z^2-(-1)\right| &= \left|z^2+1\right| \\ &= |z+i|\,|z-i| \\ &\leq \big(|z-i|+|2i|\big)\,|z-i| \\ &< 3\cdot\frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Example.

Example. $\lim_{z\to 0}\frac{\overline{z}}{z}$ *does not exist.*

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z}$$

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy}$$

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - i\lambda x}{x + i\lambda x}$$

Example. $\lim_{z\to 0} \frac{\bar{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\bar{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - i\lambda x}{x + i\lambda x} = \frac{x(1 - i\lambda)}{x(1 + i\lambda)}$$

Example. $\lim_{z\to 0}\frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - i\lambda x}{x + i\lambda x} = \frac{x(1 - i\lambda)}{x(1 + i\lambda)} = \frac{1 - i\lambda}{1 + i\lambda}$$

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - i\lambda x}{x + i\lambda x} = \frac{x(1 - i\lambda)}{x(1 + i\lambda)} = \frac{1 - i\lambda}{1 + i\lambda}$$

and the last term is independent of z.

Example. $\lim\limits_{z\to 0}\frac{\overline{z}}{z}$ *does not exist.*

For $z = x+iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x-iy}{x+iy} = \frac{x-i\lambda x}{x+i\lambda x} = \frac{x(1-i\lambda)}{x(1+i\lambda)} = \frac{1-i\lambda}{1+i\lambda}$$

and the last term is independent of z. So no matter how small we choose δ

Example. $\lim_{z\to 0} \frac{\overline{z}}{z}$ *does not exist.*

For $z = x + iy \neq 0$ with $y = \lambda x$ we have

$$\frac{\overline{z}}{z} = \frac{x - iy}{x + iy} = \frac{x - i\lambda x}{x + i\lambda x} = \frac{x(1 - i\lambda)}{x(1 + i\lambda)} = \frac{1 - i\lambda}{1 + i\lambda}$$

and the last term is independent of z. So no matter how small we choose δ, with $\lambda = \pm 1$, the output of $\frac{\overline{z}}{z}$ will not be close to any *single* number w_0.

Theorem.

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of $z_0 = x_0 + iy_0$ and let the complex function $f(z) = u(x,y) + iv(x,y)$ be defined on Ω.*

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* $z_0 = x_0 + iy_0$ *and let the complex function* $f(z) = u(x,y) + iv(x,y)$ *be defined on* Ω*. Then*

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* $z_0 = x_0 + iy_0$ *and let the complex function* $f(z) = u(x,y) + iv(x,y)$ *be defined on* Ω*. Then*

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$$

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of $z_0 = x_0 + iy_0$ and let the complex function $f(z) = u(x,y) + iv(x,y)$ be defined on Ω. Then*

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$$

and

$$\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0.$$

Proof.

Proof. First let $\lim_{z \to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$.

Proof. First let $\lim_{z \longrightarrow z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$.

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x, y) - (x_0, y_0)| < \delta$ we have

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)|$

Proof. First let $\lim_{z \to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0|$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0|$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)|$

Proof. First let $\lim_{z \to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0, y_0)| < \delta$ we have $|(x,y) - (x_0, y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$

Proof. First let $\lim_{z\longrightarrow z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.

Proof. First let $\lim_{z \to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and
$|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and
$|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$.

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and
$|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$.

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$|f(z) - (u_0 + iv_0)|$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$|f(z) - (u_0 + iv_0)| = |u(x,y) + iv(x,y) - (u_0 + iv_0)|$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$\begin{aligned} |f(z) - (u_0 + iv_0)| &= |u(x,y) + iv(x,y) - (u_0 + iv_0)| \\ &\leq |u(x,y) - u_0| \end{aligned}$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$\begin{aligned} |f(z) - (u_0 + iv_0)| &= |u(x,y) + iv(x,y) - (u_0 + iv_0)| \\ &\leq |u(x,y) - u_0| + |v(x,y) - v_0| \end{aligned}$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$\begin{aligned} |f(z) - (u_0 + iv_0)| &= |u(x,y) + iv(x,y) - (u_0 + iv_0)| \\ &\leq |u(x,y) - u_0| + |v(x,y) - v_0| < \frac{\varepsilon}{2} \end{aligned}$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$\begin{aligned} |f(z) - (u_0 + iv_0)| &= |u(x,y) + iv(x,y) - (u_0 + iv_0)| \\ &\leq |u(x,y) - u_0| + |v(x,y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z-z_0| < \delta$ we have $|f(z) - (u_0+iv_0)| < \varepsilon$. Hence for $|(x,y)-(x_0,y_0)| < \delta$ we have $|(x,y)-(x_0,y_0)| = |z-z_0| < \delta$ and then $|u(x,y)-u_0| < |f(z)-(u_0+iv_0)| < \varepsilon$ and $|v(x,y)-v_0| < |f(z)-(u_0+iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y)-(x_0,y_0)| < \delta$ we have $|u(x,y)-u_0| < \frac{\varepsilon}{2}$ and $|v(x,y)-v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y)-(x_0,y_0)| < \delta$ we have

$$\begin{aligned}
|f(z)-(u_0+iv_0)| &= |u(x,y)+iv(x,y)-(u_0+iv_0)| \\
&\leq |u(x,y)-u_0| + |v(x,y)-v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Proof. First let $\lim_{z\to z_0} f(z) = u_0 + iv_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $|f(z) - (u_0 + iv_0)| < \varepsilon$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have $|(x,y) - (x_0,y_0)| = |z - z_0| < \delta$ and then $|u(x,y) - u_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$ and $|v(x,y) - v_0| < |f(z) - (u_0 + iv_0)| < \varepsilon$.
Conversely, let $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$ and let $\varepsilon > 0$. Then there is a $\delta > 0$ so that for $|(x,y) - (x_0,y_0)| < \delta$ we have $|u(x,y) - u_0| < \frac{\varepsilon}{2}$ and $|v(x,y) - v_0| < \frac{\varepsilon}{2}$. Hence for $|(x,y) - (x_0,y_0)| < \delta$ we have

$$\begin{aligned} |f(z) - (u_0 + iv_0)| &= |u(x,y) + iv(x,y) - (u_0 + iv_0)| \\ &\leq |u(x,y) - u_0| + |v(x,y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

■

Theorem.

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex functions* f *and* F *be defined on* Ω *and so that* $\lim_{z\to z_0} f(z) = w_0$ *and* $\lim_{z\to z_0} F(z) = W_0$.

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex functions* f *and* F *be defined on* Ω *and so that* $\lim_{z\to z_0} f(z) = w_0$ *and* $\lim_{z\to z_0} F(z) = W_0$. *Then the following hold.*

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim\limits_{z\to z_0} f(z) = w_0$ and $\lim\limits_{z\to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim\limits_{z\to z_0} (f+F)(z)$

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex functions* f *and* F *be defined on* Ω *and so that* $\lim_{z\to z_0} f(z) = w_0$ *and* $\lim_{z\to z_0} F(z) = W_0$. *Then the following hold.*

1. $\lim_{z\to z_0} (f+F)(z) = w_0 + W_0$.

Theorem. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex functions* f *and* F *be defined on* Ω *and so that* $\lim_{z\to z_0} f(z) = w_0$ *and* $\lim_{z\to z_0} F(z) = W_0$*. Then the following hold.*

1. $\lim_{z\to z_0} (f+F)(z) = w_0 + W_0$.
2. $\lim_{z\to z_0} (f-F)(z)$

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim_{z \to z_0} (f + F)(z) = w_0 + W_0$.
2. $\lim_{z \to z_0} (f - F)(z) = w_0 - W_0$.

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim_{z\to z_0} (f+F)(z) = w_0 + W_0$.
2. $\lim_{z\to z_0} (f-F)(z) = w_0 - W_0$.
3. $\lim_{z\to z_0} (f\cdot F)(z)$

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim_{z \to z_0} f(z) = w_0$ and $\lim_{z \to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim_{z \to z_0} (f+F)(z) = w_0 + W_0$.
2. $\lim_{z \to z_0} (f-F)(z) = w_0 - W_0$.
3. $\lim_{z \to z_0} (f \cdot F)(z) = w_0 W_0$.

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim_{z\to z_0} (f+F)(z) = w_0 + W_0$.
2. $\lim_{z\to z_0} (f-F)(z) = w_0 - W_0$.
3. $\lim_{z\to z_0} (f\cdot F)(z) = w_0 W_0$.
4. *If* $\lim_{z\to z_0} F(z) \neq 0$, *then* $\lim_{z\to z_0} \left(\frac{f}{F}\right)(z)$

Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex functions f and F be defined on Ω and so that $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} F(z) = W_0$. Then the following hold.*

1. $\lim_{z\to z_0} (f+F)(z) = w_0 + W_0$.
2. $\lim_{z\to z_0} (f-F)(z) = w_0 - W_0$.
3. $\lim_{z\to z_0} (f\cdot F)(z) = w_0 W_0$.
4. *If* $\lim_{z\to z_0} F(z) \neq 0$, *then* $\lim_{z\to z_0} \left(\frac{f}{F}\right)(z) = \frac{w_0}{W_0}$.

Proof (products only).

Proof (products only). Let $\varepsilon > 0$.

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z-z_0| < \delta_f$ we have $\left|f(z)-w_0\right| < \dfrac{\varepsilon}{2\left(|W_0|+1\right)}$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$.

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$.

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$.

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0|+1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0|+1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\left|(f \cdot F)(z) - w_0 W_0\right|$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\left|(f \cdot F)(z) - w_0 W_0\right| \leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right|$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\begin{aligned}
\left|(f \cdot F)(z) - w_0 W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right|
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\begin{aligned}
\left|(f \cdot F)(z) - w_0 W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right| + \left|f(z) - w_0\right|\left|W_0\right|
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z-z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0|+1\right)}$ and $\left|f(z)\right| < |w_0|+1$. Moreover, there is a $\delta_F > 0$ so that for $|z-z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0|+1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z-z_0| < \delta$ we have

$$\begin{aligned}
\left|(f\cdot F)(z) - w_0W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right| + \left|f(z) - w_0\right|\left|W_0\right| \\
&< \left(|w_0|+1\right)
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\begin{aligned}
\left|(f \cdot F)(z) - w_0 W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right| + \left|f(z) - w_0\right|\left|W_0\right| \\
&< \left(|w_0| + 1\right)\frac{\varepsilon}{2\left(|w_0| + 1\right)}
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z-z_0| < \delta_f$ we have $\left|f(z)-w_0\right| < \dfrac{\varepsilon}{2\left(|W_0|+1\right)}$ and $\left|f(z)\right| < |w_0|+1$. Moreover, there is a $\delta_F > 0$ so that for $|z-z_0| < \delta_F$ we have $\left|F(z)-W_0\right| < \dfrac{\varepsilon}{2\left(|w_0|+1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z-z_0| < \delta$ we have

$$\begin{aligned}
\left|(f\cdot F)(z)-w_0W_0\right| &\leq \left|f(z)F(z)-f(z)W_0\right| + \left|f(z)W_0-w_0W_0\right| \\
&= \left|f(z)\right|\left|F(z)-W_0\right| + \left|f(z)-w_0\right|\left|W_0\right| \\
&< \left(|w_0|+1\right)\frac{\varepsilon}{2\left(|w_0|+1\right)} + \frac{\varepsilon}{2\left(|W_0|+1\right)}
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\begin{aligned}
\left|(f \cdot F)(z) - w_0 W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right| + \left|f(z) - w_0\right|\left|W_0\right| \\
&< \left(|w_0| + 1\right)\frac{\varepsilon}{2\left(|w_0| + 1\right)} + \frac{\varepsilon}{2\left(|W_0| + 1\right)}\left|W_0\right|
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\big|f(z) - w_0\big| < \dfrac{\varepsilon}{2\big(|W_0| + 1\big)}$ and $\big|f(z)\big| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\big|F(z) - W_0\big| < \dfrac{\varepsilon}{2\big(|w_0| + 1\big)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$\begin{aligned}
\big|(f \cdot F)(z) - w_0 W_0\big| &\leq \big|f(z)F(z) - f(z)W_0\big| + \big|f(z)W_0 - w_0 W_0\big| \\
&= \big|f(z)\big|\big|F(z) - W_0\big| + \big|f(z) - w_0\big|\big|W_0\big| \\
&< \big(|w_0| + 1\big)\frac{\varepsilon}{2\big(|w_0| + 1\big)} + \frac{\varepsilon}{2\big(|W_0| + 1\big)}\big|W_0\big| \\
&< \varepsilon
\end{aligned}$$

Proof (products only). Let $\varepsilon > 0$. Then there is a $\delta_f > 0$ so that for $|z - z_0| < \delta_f$ we have $\left|f(z) - w_0\right| < \dfrac{\varepsilon}{2\left(|W_0| + 1\right)}$ and $\left|f(z)\right| < |w_0| + 1$. Moreover, there is a $\delta_F > 0$ so that for $|z - z_0| < \delta_F$ we have $\left|F(z) - W_0\right| < \dfrac{\varepsilon}{2\left(|w_0| + 1\right)}$. Let $\delta := \min\{\delta_f, \delta_F\}$. Then for $|z - z_0| < \delta$ we have

$$
\begin{aligned}
\left|(f \cdot F)(z) - w_0 W_0\right| &\leq \left|f(z)F(z) - f(z)W_0\right| + \left|f(z)W_0 - w_0 W_0\right| \\
&= \left|f(z)\right|\left|F(z) - W_0\right| + \left|f(z) - w_0\right|\left|W_0\right| \\
&< \left(|w_0| + 1\right)\frac{\varepsilon}{2\left(|w_0| + 1\right)} + \frac{\varepsilon}{2\left(|W_0| + 1\right)}\left|W_0\right| \\
&< \varepsilon
\end{aligned}
$$

■

Theorem.

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$*, then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$*, then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof.

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$, *then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof.

$$\lim_{z \to z_0} P(z)$$

Theorem. *If* $P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ *and* $z_0\in\mathbb{C}$*, then* $\lim_{z\to z_0}P(z)=P(z_0)$.

Proof.

$$\lim_{z\to z_0}P(z) = \lim_{z\to z_0}a_0+a_1z+a_2z^2+\cdots+a_nz^n$$

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$, *then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof.

$$\begin{aligned} \lim_{z \to z_0} P(z) &= \lim_{z \to z_0} a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \\ &= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 z + \lim_{z \to z_0} a_2 z^2 + \cdots + \lim_{z \to z_0} a_n z^n \end{aligned}$$

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$, *then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof.

$$
\begin{aligned}
\lim_{z\to z_0} P(z) &= \lim_{z\to z_0} a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \\
&= \lim_{z\to z_0} a_0 + \lim_{z\to z_0} a_1 z + \lim_{z\to z_0} a_2 z^2 + \cdots + \lim_{z\to z_0} a_n z^n \\
&= a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n
\end{aligned}
$$

Theorem. *If* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *and* $z_0 \in \mathbb{C}$, *then* $\lim_{z \to z_0} P(z) = P(z_0)$.

Proof.

$$\begin{aligned}
\lim_{z \to z_0} P(z) &= \lim_{z \to z_0} a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \\
&= \lim_{z \to z_0} a_0 + \lim_{z \to z_0} a_1 z + \lim_{z \to z_0} a_2 z^2 + \cdots + \lim_{z \to z_0} a_n z^n \\
&= a_0 + a_1 z_0 + a_2 z_0^2 + \cdots + a_n z_0^n \\
&= P(z_0)
\end{aligned}$$

Theorem. *If* $P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ *and* $z_0\in\mathbb{C}$, *then* $\lim_{z\to z_0}P(z)=P(z_0)$.

Proof.

$$\begin{aligned}\lim_{z\to z_0}P(z) &= \lim_{z\to z_0}a_0+a_1z+a_2z^2+\cdots+a_nz^n\\ &= \lim_{z\to z_0}a_0+\lim_{z\to z_0}a_1z+\lim_{z\to z_0}a_2z^2+\cdots+\lim_{z\to z_0}a_nz^n\\ &= a_0+a_1z_0+a_2z_0^2+\cdots+a_nz_0^n\\ &= P(z_0)\end{aligned}$$

■

Observation

Observation

The function $t(z) := \frac{1}{z}$ has the following properties.

Observation

The function $t(z) := \frac{1}{z}$ has the following properties.

1. It is undefined at 0.

Observation

The function $t(z) := \frac{1}{z}$ has the following properties.

1. It is undefined at 0.
2. Circles $|z| = r$ are mapped to circles $|z| = \frac{1}{r}$

Observation

The function $t(z) := \frac{1}{z}$ has the following properties.

1. It is undefined at 0.
2. Circles $|z| = r$ are mapped to circles $|z| = \frac{1}{r}$
3. So "small" numbers are mapped to large numbers and vice versa.

Observation

The function $t(z) := \frac{1}{z}$ has the following properties.

1. It is undefined at 0.
2. Circles $|z| = r$ are mapped to circles $|z| = \frac{1}{r}$
3. So "small" numbers are mapped to large numbers and vice versa.
4. We can say that the function t "swaps 0 and ∞".

Definition.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* $w_0 \in \mathbb{C}$ *is the* **limit** *of the function* f *at* ∞

Definition. *Let $\Omega \subseteq \mathbb{C}$ contain the exterior of a disk around 0 and let the complex function f be defined on Ω. Then $w_0 \in \mathbb{C}$ is the* **limit** *of the function f at ∞ if and only if for all $\varepsilon > 0$*

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* $w_0 \in \mathbb{C}$ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω. *Then* $w_0 \in \mathbb{C}$ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* $z \in \Omega$ *with* $|z| > \frac{1}{\delta}$ *we have that* $\left|f(z) - w_0\right| < \varepsilon$.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* $w_0 \in \mathbb{C}$ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* $z \in \Omega$ *with* $|z| > \frac{1}{\delta}$ *we have that* $\left|f(z) - w_0\right| < \varepsilon$*.*

Notation. $\lim\limits_{z \to \infty} f(z) = w_0$.

Definition.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex function* f *be defined on* Ω.

Definition. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then ∞ is the* **limit** *of the function f at z_0*

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex function* f *be defined on* Ω. *Then* ∞ *is the* **limit** *of the function* f *at* z_0 *if and only if for all* $\varepsilon > 0$

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex function* f *be defined on* Ω*. Then* ∞ *is the* **limit** *of the function* f *at* z_0 *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$

Definition. *Let $\Omega \subseteq \mathbb{C}$ be a deleted neighborhood of z_0 and let the complex function f be defined on Ω. Then ∞ is the* **limit** *of the function f at z_0 if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $z \in \Omega$ with $|z - z_0| < \delta$ we have that $\left|f(z)\right| > \frac{1}{\varepsilon}$.*

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *be a deleted neighborhood of* z_0 *and let the complex function* f *be defined on* Ω. *Then* ∞ *is the* **limit** *of the function* f *at* z_0 *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* $z \in \Omega$ *with* $|z - z_0| < \delta$ *we have that* $\left|f(z)\right| > \frac{1}{\varepsilon}$.

Notation. $\lim_{z \to z_0} f(z) = \infty$.

Definition.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* ∞ *is the* **limit** *of the function* f *at* ∞

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* ∞ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$

Definition. *Let $\Omega \subseteq \mathbb{C}$ contain the exterior of a disk around 0 and let the complex function f be defined on Ω. Then ∞ is the* **limit** *of the function f at ∞ if and only if for all $\varepsilon > 0$ there is a $\delta > 0$*

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω. *Then* ∞ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* $z \in \Omega$ *with* $|z| > \frac{1}{\delta}$ *we have that* $\left|f(z)\right| > \frac{1}{\varepsilon}$.

Definition. *Let* $\Omega \subseteq \mathbb{C}$ *contain the exterior of a disk around* 0 *and let the complex function* f *be defined on* Ω*. Then* ∞ *is the* **limit** *of the function* f *at* ∞ *if and only if for all* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* $z \in \Omega$ *with* $|z| > \frac{1}{\delta}$ *we have that* $\left|f(z)\right| > \frac{1}{\varepsilon}$.

Notation. $\lim\limits_{z \to \infty} f(z) = \infty$.

Theorem.

Theorem. *Let* $z_0, w_0 \in \mathbb{C}$.

Theorem. *Let* $z_0, w_0 \in \mathbb{C}$. *Then the following hold.*

Theorem. *Let $z_0, w_0 \in \mathbb{C}$. Then the following hold.*

1. $\lim_{z\to z_0} f(z) = \infty$ *if and only if* $\lim_{z\to z_0} \frac{1}{f(z)} = 0$

Theorem. *Let $z_0, w_0 \in \mathbb{C}$. Then the following hold.*

1. $\lim_{z\to z_0} f(z) = \infty$ *if and only if* $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z\to\infty} f(z) = w_0$ *if and only if* $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$

Theorem. *Let $z_0, w_0 \in \mathbb{C}$. Then the following hold.*

1. $\lim_{z\to z_0} f(z) = \infty$ *if and only if* $\lim_{z\to z_0} \frac{1}{f(z)} = 0$
2. $\lim_{z\to\infty} f(z) = w_0$ *if and only if* $\lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$
3. $\lim_{z\to\infty} f(z) = \infty$ *if and only if* $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$

Proof.

Proof. For every statement, the key to the proof is that

Proof. For every statement, the key to the proof is that, informally speaking

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."

For example, for part 3

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.

Then for every $\varepsilon > 0$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."

For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.

Then for every $\varepsilon > 0$ there is a $\delta > 0$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$.

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so

$\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$.

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$. Thus $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$. Thus $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

The converse

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$. Thus $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$.

The converse, which says that $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$ implies $\lim_{z\to\infty} f(z) = \infty$

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$. Thus $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

The converse, which says that $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$ implies $\lim_{z\to\infty} f(z) = \infty$, is proved similarly.

Proof. For every statement, the key to the proof is that, informally speaking, $z \mapsto \frac{1}{z}$ "interchanges zero and infinity."
For example, for part 3, first consider the situation that $\lim_{z\to\infty} f(z) = \infty$.
Then for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all $|z| > \frac{1}{\delta}$ we have $\left|f(z)\right| > \frac{1}{\varepsilon}$. But then for all $|z| < \delta$ we have that $\left|\frac{1}{z}\right| > \frac{1}{\delta}$, so $\left|f\left(\frac{1}{z}\right)\right| > \frac{1}{\varepsilon}$ and hence $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \varepsilon$. Thus $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$.

The converse, which says that $\lim_{z\to 0} \dfrac{1}{f\left(\frac{1}{z}\right)} = 0$ implies $\lim_{z\to\infty} f(z) = \infty$, is proved similarly. ■

Example.

Example. $\lim_{z\to 2} \frac{z+1}{z-2}$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

Example. $\lim_{z\to 2} \frac{z+1}{z-2} = \infty$, *because* $\lim_{z\to 2} \frac{z-2}{z+1} = 0$.

$$\lim_{z\to\infty} \frac{z+1}{z-1}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}$$

Example. $\lim_{z\to 2} \frac{z+1}{z-2} = \infty$, *because* $\lim_{z\to 2} \frac{z-2}{z+1} = 0$.

$$\lim_{z\to\infty} \frac{z+1}{z-1} = \lim_{z\to 0} \frac{\frac{1}{z}+1}{\frac{1}{z}-1} = \lim_{z\to 0} \frac{\frac{1+z}{z}}{\frac{1-z}{z}}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}$$

Example. $\lim_{z\to 2} \frac{z+1}{z-2} = \infty$, *because* $\lim_{z\to 2} \frac{z-2}{z+1} = 0$.

$$\lim_{z\to\infty} \frac{z+1}{z-1} = \lim_{z\to 0} \frac{\frac{1}{z}+1}{\frac{1}{z}-1} = \lim_{z\to 0} \frac{\frac{1+z}{z}}{\frac{1-z}{z}} = \lim_{z\to 0} \frac{1+z}{1-z} = 1$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$$\lim_{z\to\infty}\frac{z^2+1}{z-1}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$$\lim_{z\to\infty}\frac{z^2+1}{z-1}=\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$$\lim_{z\to\infty}\frac{z^2+1}{z-1}=\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z^2}{z^2}}{\frac{1-z}{z}}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$$\lim_{z\to\infty}\frac{z^2+1}{z-1}=\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z^2}{z^2}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z^2}{(1-z)z}$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$$\lim_{z\to\infty}\frac{z^2+1}{z-1}=\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z^2}{z^2}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z^2}{(1-z)z}=\infty$$

Example. $\lim_{z\to 2}\frac{z+1}{z-2}=\infty$, *because* $\lim_{z\to 2}\frac{z-2}{z+1}=0$.

$$\lim_{z\to\infty}\frac{z+1}{z-1}=\lim_{z\to 0}\frac{\frac{1}{z}+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z}{z}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z}{1-z}=1$$

$\lim_{z\to\infty}\frac{z^2+1}{z-1}=\lim_{z\to 0}\frac{\left(\frac{1}{z}\right)^2+1}{\frac{1}{z}-1}=\lim_{z\to 0}\frac{\frac{1+z^2}{z^2}}{\frac{1-z}{z}}=\lim_{z\to 0}\frac{1+z^2}{(1-z)z}=\infty$, *because* $\lim_{z\to 0}\frac{(1-z)z}{1+z^2}=0$.

Definition.

Definition. *A function f is called* **continuous at** z_0 *if and only if*

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists*

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists*

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z\to z_0} f(z)$ *exists, and*
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

Definition. *A function* f *is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists, and*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if* f *is continuous at every point in its domain.*

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists, and*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem.

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z\to z_0} f(z)$ *exists, and*
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem. f *is continuous at z_0 if and only if*

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists, and*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

f is called **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem. *f is continuous at z_0 if and only if for every* $\varepsilon > 0$

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z\to z_0} f(z)$ *exists, and*
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem. f *is continuous at z_0 if and only if for every $\varepsilon > 0$, there is a* $\delta > 0$

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists, and*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem. f *is continuous at z_0 if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all z in the domain*

Definition. *A function* f *is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z\to z_0} f(z)$ *exists, and*
3. $\lim_{z\to z_0} f(z) = f(z_0)$.

f *is called* **continuous** *(on its domain) if and only if* f *is continuous at every point in its domain.*

Theorem. f *is continuous at* z_0 *if and only if for every* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that for all* z *in the domain with* $|z - z_0| < \delta$

Definition. *A function f is called* **continuous at** z_0 *if and only if*

1. $f(z_0)$ *exists, and*
2. $\lim_{z \to z_0} f(z)$ *exists, and*
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

f is called **continuous** *(on its domain) if and only if f is continuous at every point in its domain.*

Theorem. *f is continuous at z_0 if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that for all z in the domain with $|z - z_0| < \delta$ we have $\big|f(z) - f(z_0)\big| < \varepsilon$.*

Theorem.

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof.

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$.

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f\circ g$ is continuous at z_0.*

Proof. Let $\varepsilon>0$. Because f is continuous at $g(z_0)$, there is a $\delta_f>0$ so that for all y with $\left|y-g(z_0)\right|<\delta_f$ we have $\left|f(y)-f(g(z_0))\right|<\varepsilon$.

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f\circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y-g(z_0)\right| < \delta_f$ we have $\left|f(y)-f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$.

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.
Therefore we have proved that for every $\varepsilon > 0$

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.
Therefore we have proved that for every $\varepsilon > 0$ there is a $\delta > 0$

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.
Therefore we have proved that for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$

Theorem. *Let f,g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f\circ g$ is continuous at z_0.*

Proof. Let $\varepsilon>0$. Because f is continuous at $g(z_0)$, there is a $\delta_f>0$ so that for all y with $\left|y-g(z_0)\right|<\delta_f$ we have $\left|f(y)-f\left(g(z_0)\right)\right|<\varepsilon$.
Because g is continuous at z_0, for δ_f there is a $\delta>0$ so that for all z with $|z-z_0|<\delta$ we have $\left|g(z)-g(z_0)\right|<\delta_f$. But then because $\left|g(z)-g(z_0)\right|<\delta_f$ we infer $\left|f\left(g(z)\right)-f\left(g(z_0)\right)\right|<\varepsilon$.
Therefore we have proved that for every $\varepsilon>0$ there is a $\delta>0$ so that for all z with $|z-z_0|<\delta$ we have $\left|f\left(g(z)\right)-f\left(g(z_0)\right)\right|<\varepsilon$

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.
Therefore we have proved that for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$, which means that $f \circ g$ is continuous at z_0.

Theorem. *Let f, g be functions so that z_0 is in the (open) domain of g and $g(z_0)$ is in the (open) domain of f. If g is continuous at z_0 and f is continuous at $g(z_0)$, then $f \circ g$ is continuous at z_0.*

Proof. Let $\varepsilon > 0$. Because f is continuous at $g(z_0)$, there is a $\delta_f > 0$ so that for all y with $\left|y - g(z_0)\right| < \delta_f$ we have $\left|f(y) - f(g(z_0))\right| < \varepsilon$. Because g is continuous at z_0, for δ_f there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|g(z) - g(z_0)\right| < \delta_f$. But then because $\left|g(z) - g(z_0)\right| < \delta_f$ we infer $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$.
Therefore we have proved that for every $\varepsilon > 0$ there is a $\delta > 0$ so that for all z with $|z - z_0| < \delta$ we have $\left|f(g(z)) - f(g(z_0))\right| < \varepsilon$, which means that $f \circ g$ is continuous at z_0. ■

Theorem.

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof.

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := |f(z_0)|$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$.

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\left|f(z)\right|$$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \big|f(z_0)\big|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\big|f(z) - f(z_0)\big| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\big|f(z)\big| = \big|f(z_0) - \big(f(z_0) - f(z)\big)\big|$$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\left|f(z)\right| = \left|f(z_0) - \left(f(z_0) - f(z)\right)\right| \geq \left|f(z_0)\right| - \left|f(z_0) - f(z)\right|$$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\begin{aligned} \left|f(z)\right| &= \left|f(z_0) - (f(z_0) - f(z))\right| \geq \left|f(z_0)\right| - \left|f(z_0) - f(z)\right| \\ &> \left|f(z_0)\right| - \left|f(z_0)\right| \end{aligned}$$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\begin{aligned} \left|f(z)\right| &= \left|f(z_0) - \left(f(z_0) - f(z)\right)\right| \geq \left|f(z_0)\right| - \left|f(z_0) - f(z)\right| \\ &> \left|f(z_0)\right| - \left|f(z_0)\right| = 0 \end{aligned}$$

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\begin{aligned} \left|f(z)\right| &= \left|f(z_0) - \big(f(z_0) - f(z)\big)\right| \geq \left|f(z_0)\right| - \left|f(z_0) - f(z)\right| \\ &> \left|f(z_0)\right| - \left|f(z_0)\right| = 0, \end{aligned}$$

which means that $f(z) \neq 0$ for $|z - z_0| < \delta$.

Theorem. *If f is continuous at z_0 and $f(z_0) \neq 0$, then there is a neighborhood of z_0 on which f has no zeros.*

Proof. For $\varepsilon := \left|f(z_0)\right|$ there is a $\delta > 0$ so that for $|z - z_0| < \delta$ we have $\left|f(z) - f(z_0)\right| < \varepsilon$. Therefore, for $|z - z_0| < \delta$ we have

$$\begin{aligned} \left|f(z)\right| &= \left|f(z_0) - (f(z_0) - f(z))\right| \geq \left|f(z_0)\right| - \left|f(z_0) - f(z)\right| \\ &> \left|f(z_0)\right| - \left|f(z_0)\right| = 0, \end{aligned}$$

which means that $f(z) \neq 0$ for $|z - z_0| < \delta$. ■